ANALYTICITY OF THE SCATTERING OPERATOR FOR SEMILINEAR DISPERSIVE EQUATIONS

RÉMI CARLES AND ISABELLE GALLAGHER

Abstract. We present a general algorithm to show that a scattering operator associated to a semilinear dispersive equation is real analytic, and to compute the coefficients of its Taylor series at any point. We illustrate this method in the case of the Schrödinger equation with power-like nonlinearity or with Hartree type nonlinearity, and in the case of the wave and Klein–Gordon equations with power nonlinearity. Finally, we discuss the link of this approach with inverse scattering, and with complete integrability.

1. Introduction

The local and global well-posedness of semilinear dispersive equations has attracted a lot of attention for the past years. In general, when global well-posedness is established, the existence of a scattering operator, comparing the nonlinear dynamics and the linear one, is a rather direct by-product. Unlike in the linear case (see e.g. [45, 56, 64]), besides continuity, very few properties of these nonlinear scattering operators are known. A first natural question, which can be found in [55, pp. 121–122], consists in investigating the real analyticity of the scattering operators. A positive answer is available in some very specific cases: see [7, 8, 44] for the cubic wave and Klein–Gordon equation in 3D, and [48] for the Hartree equation in 3D. In this paper, we extend these results to a more general class of dispersive equations, including the nonlinear Schrödinger equation and the nonlinear wave equation, in space dimension $n \leqslant 4$ (such an assumption is needed for the power nonlinearity to be both analytic and energy-subcritical or critical). Moreover, unlike in [7, 8, 44, 48], we do not use an abstract analytic implicit function theorem: we construct directly the terms of the series *via* a general abstract lemma, thus extending the approach of S. Masaki [47]. We then show that the series is converging, working in suitable spaces based on dispersive properties provided by Strichartz estimates. In general, these estimates are a direct by-product of the proof of the existence of a nonlinear scattering operator.

Before being more precise about the results presented here, we briefly recall the approach for (short range) scattering theory in the context of semilinear dispersive equations. The main examples we have in mind are

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the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

the Hartree equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \left(|x|^{-\gamma} * |u|^2\right) u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.2}$$

the nonlinear wave and Klein–Gordon equations

$$\partial_t^2 u - \Delta u + \lambda u^p = 0, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n \tag{1.3}$$

$$\partial_t^2 u - \Delta u + u + \lambda u^p = 0, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n. \tag{1.4}$$

Up to considering the unknown $(u, \partial_t u)$ instead of u alone in (1.3), (1.4), Duhamel's formula reads, in all these examples,

$$u(t) = U(t)u_0 + \int_{t_0}^t U(t-s)\left(F(u(s))\,ds,\right. \tag{1.5}$$

where $U(\cdot)$ is the group associated to the linear equation ($\lambda = 0$), and t_0 corresponds to the time for which initial data are prescribed:

$$U(-t)u(t)\big|_{t=t_0} = u_0. \tag{1.6}$$

In the study of the Cauchy problem, one usually considers the case $t_0 = 0$. In scattering theory, the first standard step consists in solving the Cauchy problem near infinite time: $t_0 = \pm\infty$. To consider forward in time propagation, assume $t_0 = -\infty$. To define the wave operator W_-, one has to solve the Cauchy problem (1.5)-(1.6) with $t_0 = -\infty$, on some time interval of the form $]-\infty, T]$, for some finite T. Classically, this step is achieved by a fixed point argument in suitable function spaces. This may yield a time $T \ll -1$, that is, "close" to $-\infty$ (but finite). Suppose that the classical Cauchy problem enables us to define u up to time $t = 0$. Then the wave operator W_- is defined by

$$W_- u_0 = u_{|t=0}.$$

The second step consists in inverting the wave operators. For initial data prescribed at time $t = 0$, suppose that we can construct a solution which is defined globally in time (or in the future only, for our purpose). Inverting the wave operators (that is, proving the asymptotic completeness) consists in showing that nonlinear effects become negligible for large time, and that we can find u_+ such that $u(t) \sim U(t)u_+$ as $t \to +\infty$:

$$u_+ = W_+^{-1} u_{|t=0}.$$

The scattering operator S is then defined by

$$Su_0 = W_+^{-1} W_- u_0 = u_+.$$

In general, for small data, the scattering operator S can be constructed in one step only, thanks to a bootstrap argument in spaces based on Strichartz estimates. For large data, one must expect $T \ll -1$ in general. The solution is then made global thanks to *a priori* estimates, such as the conservation of a positive energy ($\lambda > 0$ in the above examples). The proof of asymptotic completeness usually relies on different arguments: Morawetz estimates, or existence of an extra evolution law (*e.g.* pseudo-conformal evolution law).

In many cases, these arguments make it possible to define the scattering operator. The continuity of this operator is usually an easy consequence of its construction (provided that the proof does not rely on compactness arguments). Finer properties, such as real analyticity, are not straightforward. We emphasize again that contrary to the case of the wave operators, real analyticity of the scattering operator (for arbitrary data) cannot be a mere consequence of the fixed point method used to construct solutions; we show here that real analyticity of the scattering operator is very often a consequence of the (global in time) estimates which are established in order to show that there is scattering. In all this paper, by "analytic", we mean "real analytic":

Definition 1.1. *Let X and Y be Banach spaces, and consider an operator $A : X \to Y$. We say that A is real analytic (or simply analytic) from X to Y if A is infinitely Fréchet-differentiable at every point of X, with a locally norm-convergent series: for all $f \in X$, there exists $\varepsilon_0 > 0$, such that for all $g \in X$, $\|g\|_X \leqslant 1$, we can find $(w_j)_{j\in\mathbb{N}} \in Y^{\mathbb{N}}$ such that for $0 < \varepsilon \leqslant \varepsilon_0$,*

$$\sum_{j=0}^{\infty} \varepsilon^j \|w_j\|_Y < \infty, \quad \textit{and} \quad A(f+\varepsilon g) = A(f) + \varepsilon \sum_{j=0}^{\infty} \varepsilon^j w_j .$$

First, it should be noted that the analyticity of scattering operators near the origin can be obtained rather directly in general, by applying a fixed point argument with analytic parameters. Of course, if the nonlinearity is not analytic, one must not expect the scattering operator to be analytic. As an illustration, consider the nonlinear Schrödinger equation (1.1). As noticed in [17] (in the case $n = 1$), and following the approach of [25], the first terms of the asymptotic expansion of the nonlinear scattering operator S near the origin are given by:

$$S(\varepsilon u_-) = \varepsilon u_- - i\varepsilon^p \int_{-\infty}^{+\infty} e^{-i\frac{t}{2}\Delta}\left(\left| e^{i\frac{t}{2}\Delta} u_- \right|^{p-1} e^{i\frac{t}{2}\Delta} u_- \right) dt + \mathcal{O}_{L^2}\left(\varepsilon^{2p-1}\right).$$

The complete proof of this relation is available in [18] in the L^2-critical case $p = 1 + 4/n$, for any $n \geqslant 1$. This shows that if p is not an integer, the operator S is not analytic near the origin: it is Hölder continuous, of order p and not better. We shall therefore consider only analytic nonlinearities: in (1.3), (1.4), we shall always assume that p is an integer, and in (1.1), we shall assume that p is an odd integer. We can now state two typical results of our approach. Denote

$$\Sigma = \{ f \in H^1(\mathbb{R}^n), \quad x \mapsto |x| f(x) \in L^2(\mathbb{R}^n) \}.$$

This space is naturally a Hilbert space. The main results of the paper are the following.

Theorem 1.2. *Let $1 \leqslant n \leqslant 4$ and $\lambda > 0$. Assume that $p \geqslant 3$ is an odd integer, with in addition*

- *$p \geqslant 5$ if $n = 1$.*
- *$p = 3$ or 5 if $n = 3$.*
- *$p = 3$ if $n = 4$.*

Then the wave and scattering operators associated to the nonlinear Schrödinger equation (1.1) *are analytic from* Σ *to* Σ*. If moreover* $p \geqslant 7$ *for* $n = 1$ *and* $p \geqslant 5$ *for* $n = 2$*, then the wave and scattering operators associated to* (1.1) *are analytic from* $H^1(\mathbb{R}^n)$ *to* $H^1(\mathbb{R}^n)$*.*

Theorem 1.3. *Let* $n \geqslant 3$ *and* $\lambda > 0$*. Assume that* $2 \leqslant \gamma < \min(4, n)$*. Then the wave and scattering operators associated to the Hartree equation* (1.2) *are analytic from* Σ *to* Σ*. If moreover* $\gamma > 2$*, then the wave and scattering operators associated to* (1.2) *are analytic from* $H^1(\mathbb{R}^n)$ *to* $H^1(\mathbb{R}^n)$*.*

Theorem 1.4. *Let* $\lambda > 0$*. Assume that either* $(n, p) = (3, 5)$ *or* $(n, p) = (4, 3)$*. Then the wave and scattering operators associated to the nonlinear wave equation* (1.3) *are analytic* $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *to* $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*.*

Theorem 1.5. *Let* $1 \leqslant n \leqslant 4$ *and* $\lambda > 0$*. Assume that* $p \geqslant 3$ *is an odd integer, with*

- $p \geqslant 7$ *if* $n = 1$*.*
- $p \geqslant 5$ *if* $n = 2$*.*
- $p = 3$ *or* 5 *if* $n = 3$*.*
- $p = 3$ *if* $n = 4$*.*

The wave and scattering operators associated to the nonlinear Klein–Gordon equation (1.4) *are analytic from* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *to* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*.*

Notation. If A and B are two real numbers, we will write $A \lesssim B$ if there is a universal constant C, which does not depend on varying parameters of the problem, such that $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we will write $A \sim B$.

2. An abstract result

In this section we intend to study an abstract semilinear equation, and to present the assumptions we will make in order to conclude to the analyticity of the nonlinear scattering operator associated to the equation. We begin (in Section 2.1) by writing down in an informal way the equations and the expected expansion of the solution around a given state. That will motivate the computations of Section 2.2 in which an abstract result is proved, showing under what assumptions on the equation one can justify such an expansion.

2.1. **Setting of the problem.** Consider a first order partial differential equation, of the form

$$\partial_t u = L(\partial_x)u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \quad u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \text{ or } \mathbb{R}^d,\ d \geqslant 1.$$

We assume that the evolution of the solution to this linear equation is described by a group $U(t)$. In the semilinear equations we have in mind, the nonlinearity will be a power law Φ of degree $p \geqslant 2$. Let us consider any solution $\underline{u}$ to the following equation

$$\partial_t \underline{u} = L(\partial_x)\underline{u} + \Phi(\underline{u}).$$

Introduce the Duhamel formula associated to this equation:

$$\underline{u}(t) = U(t)\underline{u}_0 + N(\underline{u})(t), \tag{2.1}$$

where we have defined

$$N(\underline{u})(t) := \int_{t_0}^{t} U(t-s)\Phi\left(\underline{u}(s)\right)\, ds. \tag{2.2}$$

In scattering theory, one must think of the initial time as being infinite, $t_0 = -\infty$, in which case $\underline{u}_0 = u_-$ is an asymptotic state.

Example 2.1. To make our discussion a little more concrete, we illustrate it with the case of a nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{p-1}u. \tag{2.3}$$

In this case, $U(t) = e^{i\frac{t}{2}\Delta}$, and $\Phi(u) = -i|u|^{p-1}u$.

Example 2.2. In the case of the nonlinear wave equation

$$\partial_t^2 u - \Delta u + u^p = 0, \tag{2.4}$$

we set $\underline{u} = {}^t(u, \partial_t u)$. Denote

$$\omega = (-\Delta)^{1/2} \ ; \ W(t) = \omega^{-1}\sin(\omega t) \ ; \ \dot{W}(t) = \cos(\omega t)\,.$$

Then (2.4) takes the form (2.1)–(2.2), with

$$U(t) = \begin{pmatrix} \dot{W}(t) & W(t) \\ -\omega^2 W(t) & \dot{W}(t) \end{pmatrix} \quad ; \quad \Phi(\underline{u}) = \begin{pmatrix} 0 \\ -\underline{u}_1^p \end{pmatrix} = \begin{pmatrix} 0 \\ -u^p \end{pmatrix}.$$

The same holds in the case of the nonlinear Klein–Gordon equation

$$\partial_t^2 u - \Delta u + u + u^p = 0.$$

The only adaptation needed in this case consists in substituting ω with $\Lambda = (1-\Delta)^{1/2}$.

We suppose that this semilinear equation has global solutions in time and that a nonlinear scattering theory is available (examples are provided in Section 3 below). The discussion that follows is purely formal, and is intended as a motivation to the computations carried out in the coming paragraph.

Let us construct a solution to the equation associated with an initial data which is a perturbation of $\underline{u}_0$, written $\underline{u}_0 + \varepsilon u_0$ where ε is a small parameter, and let us write the solution u^ε under the form $u^\varepsilon = \underline{u} + w^\varepsilon$. We are looking for an expansion of the perturbation w^ε in powers of ε. Writing $\Phi(\underline{u}+w^\varepsilon)$ in terms of $\Phi(\underline{u})$ using Taylor's formula yields easily that the equation on w^ε must be of the following type:

$$w^\varepsilon(t) = U(t)(\varepsilon u_0) + \sum_{j=1}^{p} \int_{t_0}^{t} U\left(t-s\right)\Phi_j\left(\underline{u}(s), w^\varepsilon(s), \dots, w^\varepsilon(s)\right) ds, \tag{2.5}$$

where from now on $\Phi_j(\alpha_0, \alpha_1, \dots, \alpha_j)$ denotes a multi-linear form, which is $(p-j)$-linear in α_0 and linear in its j last arguments. In general, this multilinearity is on $\mathbb{R}$ only, since in the case of the nonlinear Schrödinger equation, conjugation is involved in the above formula. To ease the notations, we introduce

$$N_j(\underline{u}, w, \dots, w)(t) = \int_{t_0}^{t} U\left(t-s\right)\Phi_j\left(\underline{u}(s), w(s), \dots, w(s)\right) ds. \tag{2.6}$$

Our aim is now to write an expansion of w^ε in powers of ε, $w^\varepsilon = \sum_{k\in\mathbb{N}} \varepsilon^{k+1} w_k$. Two different situations can occur, according to the value of $\underline{u}_0$: either $\underline{u}_0$ is identically zero (and the situation corresponds to the case of small data), or it is not.

Case 1: Expansion around zero. Suppose $\underline{u}_0$ vanishes identically. In that case the only Φ_j in (2.5) which is not identically zero is when $j = p$, and each w_k can be computed explicitly: the only non vanishing terms in the expansion are terms of the type $w_{k(p-1)}$, for $k \in \mathbb{N}$, with

$$w_0(t) = U(t)u_0,$$

and where the other terms of the expansion are given by an explicit algorithm, of the form

$$w_{(k+1)(p-1)}(t) = G_k\left(w_0(t), w_{p-1}(t), \dots, w_{k(p-1)}(t)\right), \quad k \geqslant 0.$$

Typically, w_0 and w_{p-1} are given by

$$w_0(t) = U(t)u_0 \quad ; \quad w_{p-1}(t) = N_p\left(w_0(t), \dots, w_0(t)\right).$$

Example 2.3. In the above example of the nonlinear Schrödinger equation (2.3), this yields

$$\begin{aligned}
w_0(t,x) &= e^{i\frac{t}{2}\Delta} u_0(x),\\
w_{p-1}(t,x) &= -i\int_{-\infty}^{t} e^{i\frac{t-s}{2}\Delta}\left(|e^{i\frac{s}{2}\Delta}u_0(x)|^{p-1} e^{i\frac{s}{2}\Delta}u_0(x)\right) ds.
\end{aligned}$$

In other words, w_0 and w_{p-1} solve

$$\begin{aligned}
&i\partial_t w_0 + \frac{1}{2}\Delta w_0 = 0 &&; \quad U(-t)w_0(t)\big|_{t=-\infty} = u_0.\\
&i\partial_t w_{p-1} + \frac{1}{2}\Delta w_{p-1} = |w_0|^{p-1}w_0 &&; \quad U(-t)w_1(t)\big|_{t=-\infty} = 0.
\end{aligned}$$

It is obvious that $e^{-i\frac{t}{2}\Delta}w_0(t,x)$ converges as $t \to +\infty$, and part of the game consists in showing that so does $e^{-i\frac{t}{2}\Delta}w_1(t,x)$.

Case 2: Expansion around any initial data. In that case all the Φ_j's have to be taken into account in (2.5), so the series will be full if $\underline{u} \neq 0$. Moreover the w_k's are not computed explicitly. For instance the first two terms w_0 and w_1 of the expansion satisfy

$$w_0(t) = U(t)u_0 + N_1\left(\underline{u}, w_0\right)(t) \;\; ; \;\; w_1(t) = N_1\left(\underline{u}, w_1\right)(t) + N_2\left(\underline{u}, w_0, w_0\right)(t).$$

Example 2.4. In our Schrödinger example (2.3), this means that w_0 must solve

$$\begin{aligned}
&i\partial_t w_0 + \frac{1}{2}\Delta w_0 = p|\underline{u}|^{p-1}w_0 + (p-1)\underline{u}^{(p+1)/2}\overline{\underline{u}}^{(p-1)/2}\overline{w}_0,\\
&U(-t)w_0(t)\big|_{t=-\infty} = u_0.
\end{aligned}$$

Note that the above Hamiltonian is not self-adjoint in general. However, this aspect will not be an obstruction to our analysis.

Conclusion. To summarize the above considerations, the solution to the equation

$$u^\varepsilon(t) = U(t)(\varepsilon u_0) + N\left(u^\varepsilon\right)(t)$$

can be expanded as

$$u^\varepsilon = \varepsilon \sum_{k=0}^{\infty} \varepsilon^{k(p-1)} w_{k(p-1)},$$

where the $w_{k(p-1)}$ satisfy linear equations and can be computed explicitly by induction. On the other hand, the solution to the equation

$$u^\varepsilon(t) = U(t)(\underline{u}_0 + \varepsilon u_0) + N\left(u^\varepsilon\right)(t)$$

can be expanded as

$$u^\varepsilon = \underline{u} + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k w_k,$$

where again the w_k satisfy linear equations, but this time are only known implicitly (again by induction). Those expansions allow to conclude that the scattering operator is analytic, around any given state (small or large). In order to make those heuristical remarks rigorous, we need to prove the convergence of the series formally obtained above. This is performed in the next section, where we prove an abstract result stating under what conditions the series does converge.

2.2. **An abstract lemma.** In this section we adapt [47, Theorem 3.2] to the case of a perturbation around any given state (in [47], the perturbation is around zero only).

We keep the notation of the previous paragraph. Let us define D as the Banach space in which the data lies, and F the space in which the linear flow transports the data. The space F is a space-time Banach space, which we will write as $F = F_1 \cap F_2$, where

$$F_1 := (C \cap L^\infty)(\mathbb{R}; D)$$

corresponds to the energy space, while

$$F_2 = L^{q_1}(\mathbb{R}; X_1) \cap L^{q_2}(\mathbb{R}; X_2), \quad 1 \leqslant q_1, q_2 < \infty.$$

for some Banach spaces X_1 and X_2. Typically F_2 should be thought of as a Strichartz space, taking into account dispersive effects. In several applications, we will consider $q_1 = q_2$ and $X_1 = X_2$. The main assumption on the linear evolution is that

Assumption (H1). There exists $C_0 > 0$ such that for all $g \in D$,

$$\|U(\cdot)g\|_F \leqslant C_0 \|g\|_D.$$

This assumption will always be satisfied thanks to Strichartz estimates.

Example 2.5. Suppose that we consider the nonlinear Schrödinger at the L^2 level. A natural choice is then $D = L^2(\mathbb{R}^n)$, $F = (C \cap L^\infty)(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^q(\mathbb{R}; L^r(\mathbb{R}^n))$ for some Strichartz admissible pair (q, r) (with $r = p + 1$).

As in the previous paragraph we consider a family of p-linear forms denoted by $(N_j)_{1\leqslant j\leqslant p}$, who are $(p-j)$-linear in the first variable and linear in each of the j remaining variables. We recall that the family $(N_j)_{1\leqslant j\leqslant p}$ is constructed as follows:

$$\forall (a,b), \quad N(a+b)-N(b)=\sum_{j=1}^{p} N_j(a,b,\dots,b). \tag{2.7}$$

We will consider the second assumption:

Assumption (H2). There exists $\delta, C>0$ such that for all $\underline{u}, u_1,\dots,u_j \in F$ and for all I interval in $\mathbb{R}$, we have:

$$\|\mathbf{1}_{t\in I} N_j(\underline{u},u_1,\dots,u_j)\|_F \leqslant C\|\mathbf{1}_{t\in I}\underline{u}\|_{F_2}^{\delta}\|\underline{u}\|_F^{p-\delta-j}\prod_{\ell=1}^{j}\|u_\ell\|_F \text{ if } j\leqslant p-1,$$

$$\|\mathbf{1}_{t\in I} N_p(u_1,\dots,u_p)\|_F \leqslant C\sum_{\ell=1}^{p}\|\mathbf{1}_{t\in I}u_\ell\|_{F_2}^{\delta}\|u_\ell\|_F^{1-\delta}\prod_{\ell'\neq \ell}^{p}\|u_{\ell'}\|_F.$$

Remark 2.6. The definition of F implies that if A and B are two disjoint intervals of $\mathbb{R}$, then

$$\|\mathbf{1}_{t\in A\cup B} f\|_F \sim \|\mathbf{1}_{t\in A} f\|_F + \|\mathbf{1}_{t\in B} f\|_F. \tag{2.8}$$

Moreover Lebesgue's theorem implies that

$$\forall v\in F_2, \quad \lim_{T\to+\infty}\|\mathbf{1}_{t\geqslant T} v\|_{F_2}=0. \tag{2.9}$$

Similarly, we notice that $(H2)$, applied to $j=1$, implies that $\mathbb{R}$ may be decomposed into a finite, disjoint union of K intervals $(I_k)_{1\leqslant k\leqslant K}$ such that

$$\forall \underline{u}, v\in F, \quad \|\mathbf{1}_{t\in I_k} N_1(\underline{u},v)\|_F \leqslant \frac{1}{2}\|\mathbf{1}_{t\in I_k} v\|_F. \tag{2.10}$$

Fix u_0 in D. We construct by induction a family $(w_k)_{k\in\mathbb{N}}$:

$$w_0(t)=U(t)u_0+N_1(\underline{u},w_0)(t),$$

$$w_m=\sum_{j=1}^{p}\sum_{\substack{j+\ell_1+\dots+\ell_j=m+1\\ \ell_i\geqslant 0}} N_j(\underline{u},w_{\ell_1},\dots,w_{\ell_j}),$$

with the convention that $\sum_{\emptyset}=0$. We have the following important lemma.

Lemma 2.7. *Let* $\underline{u}\in F$ *solve* (2.1) *with initial data* $\underline{u}_0\in D$, *and let* u_0 *be a given function in* D, *with* $\|u_0\|_D\leqslant M$. *Assume (H1) and (H2) hold. Then there exists* $\varepsilon_0=\varepsilon_0(\|\underline{u}\|_F, M)>0$ *such that for* $0<\varepsilon\leqslant\varepsilon_0$, *the series* $\sum_{k\in\mathbb{N}}\varepsilon^k w_k$ *converges normally in* F, *and*

$$u^\varepsilon := \underline{u}+\varepsilon\sum_{k\in\mathbb{N}}\varepsilon^k w_k \quad \textit{solves:} \quad u^\varepsilon(t)=U(t)(\underline{u}_0+\varepsilon u_0)+N(u^\varepsilon)(t).$$

Remark 2.8. Lemma 2.7 implies in particular the real analyticity of the wave operators as functions of D, by considering the above result at time $t=0$, since for $t_0=-\infty$, $u^\varepsilon_{|t=0}=W_-(\underline{u}_0+\varepsilon u_0)$.

Proof. Let us start by finding a bound on w_0 in F. Inequality (2.10) allows to write that

$$\begin{aligned}\|\mathbf{1}_{t\in I_k} w_0\|_F &\leqslant \|\mathbf{1}_{t\in I_k} U(\cdot)u_0\|_F + \|\mathbf{1}_{t\in I_k} N_1(\underline{u}, w_0)\|_F \\ &\leqslant \|\mathbf{1}_{t\in I_k} U(\cdot)u_0\|_F + \frac{1}{2}\|\mathbf{1}_{t\in I_k} w_0\|_F.\end{aligned}$$

This implies directly, using (2.8), that

$$\|w_0\|_F \lesssim \|U(\cdot)u_0\|_F$$

so by $(H1)$ we infer that

$$\|w_0\|_F \lesssim C_0\|u_0\|_D. \tag{2.11}$$

We prove by induction that there exists $\Lambda \geqslant 1$ such that for all $m \geqslant 1$,

$$\|w_m\|_F \leqslant \Lambda^m. \tag{R$_m$}$$

We notice that if that is the case, then the convergence of the series $\sum_{k\in\mathbb{N}} \varepsilon^k w_k$ in F is obvious as soon as $\varepsilon\Lambda < 1$.

Let us start by proving (R_1). We have by definition

$$w_1 = N_1(\underline{u}, w_1) + N_2(\underline{u}, w_0, w_0),$$

and the same argument as in the case of w_0 gives

$$\begin{aligned}\|\mathbf{1}_{t\in I_k} w_1\|_F &\leqslant \|\mathbf{1}_{t\in I_k} N_1(\underline{u}, w_1)\|_F + \|\mathbf{1}_{t\in I_k} N_2(\underline{u}, w_0, w_0)\|_F \\ &\leqslant \frac{1}{2}\|\mathbf{1}_{t\in I_k} w_1\|_F + \|\mathbf{1}_{t\in I_k} N_2(\underline{u}, w_0, w_0)\|_F.\end{aligned}$$

By (2.8), we infer that

$$\|w_1\|_F \lesssim \|N_2(\underline{u}, w_0, w_0)\|_F.$$

The continuity property $(H2)$ then implies that

$$\|w_1\|_F \lesssim C_2\|\underline{u}\|_F^{p-2}\|w_0\|_F^2$$

so finally by (2.11)

$$\|w_1\|_F \lesssim C_2\|\underline{u}\|_F^{p-2}(C_0\|u_0\|_D)^2.$$

So we can choose $\Lambda \gtrsim 1 + C_2\|\underline{u}\|_F^{p-2}(C_0\|u_0\|_D)^2$ to get

$$\|w_1\|_F \leqslant \Lambda.$$

Now let us turn to the hierarchy of equations on w_m, for $m \geqslant 2$. Supposing that (R_ℓ) holds for all $1 \leqslant \ell \leqslant m-1$, let us prove (R_m). To simplify the notation we define

$$\widetilde{N}(\underline{u}, w_0, \dots, w_{m-1}) := \sum_{j=2}^{p} \sum_{\substack{j+\ell_1+\cdots+\ell_j=m+1 \\ \ell_i\geqslant 0}} N_j(\underline{u}, w_{\ell_1}, \dots, w_{\ell_j}).$$

The same argument as above yields

$$\begin{aligned}\|\mathbf{1}_{t\in I_k} w_m\|_F &\leqslant \|\mathbf{1}_{t\in I_k} N_1(\underline{u}, w_m)\|_F + \|\mathbf{1}_{t\in I_k} \widetilde{N}(\underline{u}, w_0, \dots, w_{m-1})\|_F \\ &\leqslant \frac{1}{2}\|\mathbf{1}_{t\in I_k} w_m\|_F + \|\mathbf{1}_{t\in I_k} \widetilde{N}(\underline{u}, w_0, \dots, w_{m-1})\|_F.\end{aligned}$$

Obviously this implies, using (2.8), that

$$\|w_m\|_F \lesssim \|\widetilde{N}(\underline{u}, w_0, \dots, w_{m-1})\|_F.$$

By $(H2)$ and defining $C := \max_{1\leqslant j\leqslant p} C_j$, we get that

$$\begin{aligned}
\|w_m\|_F &\lesssim C\sum_{j=2}^{p} \|\underline{u}\|_F^{p-j} \sum_{\substack{j+\ell_1+\cdots+\ell_j=m+1\\ \ell_i\geqslant 0}} \prod_{i=1}^{j} \|w_{\ell_i}\|_F\\
&\lesssim C\sum_{j=2}^{p} \|\underline{u}\|_F^{p-j} \sum_{\substack{j+\ell_1+\cdots+\ell_j=m+1\\ \ell_i\geqslant 0}} \prod_{i=1}^{j} \Lambda^{\ell_i}(C_0\|u_0\|_D)^{\sharp\{i,\ell_i=0\}}\\
&\lesssim C(1+C_0\|u_0\|_D+\|\underline{u}\|_F)^p\sum_{j=2}^{p}\Lambda^{m+1-j}\\
&\lesssim C(1+C_0\|u_0\|_D+\|\underline{u}\|_F)^p\Lambda^{m-1}
\end{aligned}$$

since $\Lambda \geqslant 1$. To summarize, choosing

$$\Lambda \gtrsim 1 + C(1+C_0\|u_0\|_D+\|\underline{u}\|_F)^p$$

we have $\|w_m\|_F \leqslant \Lambda^m$, and (R_m) is proved for all $m \geqslant 1$. As remarked above, this enables us to infer that as soon as ε is small enough, the series of general term $\varepsilon^k w_k$ is convergent.

To conclude the proof of the lemma, let us prove that the solution of

$$u^\varepsilon(t) = U(t)(\underline{u}_0 + \varepsilon u_0) + N\left(u^\varepsilon\right)(t) \tag{2.12}$$

satisfies

$$u^\varepsilon = \underline{u} + \varepsilon \sum_{k\in\mathbb{N}} \varepsilon^k w_k.$$

We show that the solution u^ε of (2.12) satisfies

$$\lim_{n\to\infty}\left\| u^\varepsilon - \underline{u} - \varepsilon\sum_{k=0}^{n}\varepsilon^k w_k\right\|_F = 0,$$

by writing the equation satisfied by $\widetilde{w}_n^\varepsilon := u^\varepsilon - \underline{u} - \varepsilon\sum_{k=0}^n \varepsilon^k w_k$. It is here that the exact definition of the multi-linear operators N_j given in (2.7) is used. First, we know that for $\varepsilon\Lambda < 1$, the series $\sum \varepsilon^k w_k$ converges normally in F. Therefore, $\widetilde{w}_n^\varepsilon$ has a limit in F as $n \to \infty$, provided that ε is fixed such that $\varepsilon\Lambda < 1$. On the other hand, by the definition of $\widetilde{w}_n^\varepsilon$,

$$\begin{aligned}\widetilde{w}_n^\varepsilon &= N\left(\underline{u} + \varepsilon\sum_{k=0}^{n}\varepsilon^k w_k + \widetilde{w}_n^\varepsilon\right) - N(\underline{u})\\ &\quad - \varepsilon\sum_{k=0}^{n}\varepsilon^k\sum_{j=1}^{p}\sum_{\substack{j+\ell_1+\dots+\ell_j=k+1\\ \ell_i\geqslant 0}} N_j(\underline{u}, w_{\ell_1},\dots,w_{\ell_j})\\ &= \sum_{j=1}^{p} N_j\left(\underline{u}, \varepsilon\sum_{\ell_1=0}^{n}\varepsilon^{\ell_1}w_{\ell_1} + \widetilde{w}_n^\varepsilon, \dots, \varepsilon\sum_{\ell_j=0}^{n}\varepsilon^{\ell_j}w_{\ell_j} + \widetilde{w}_n^\varepsilon\right)\\ &\quad - \varepsilon\sum_{k=0}^{n}\varepsilon^k\sum_{j=1}^{p}\sum_{\substack{j+\ell_1+\dots+\ell_j=k+1\\ \ell_i\geqslant 0}} N_j(\underline{u}, w_{\ell_1},\dots,w_{\ell_j})\end{aligned}$$

From the above estimates, we can write

$$\begin{aligned}\sum_{j=1}^{p} N_j\left(\underline{u}, \varepsilon\sum_{\ell_1=0}^{n}\varepsilon^{\ell_1}w_{\ell_1} + \widetilde{w}_n^\varepsilon, \dots, \varepsilon\sum_{\ell_j=0}^{n}\varepsilon^{\ell_j}w_{\ell_j} + \widetilde{w}_n^\varepsilon\right) &= G_n\left(\widetilde{w}_n^\varepsilon\right)\\ + \sum_{j=1}^{p} N_j\left(\underline{u}, \varepsilon\sum_{\ell_1=0}^{n}\varepsilon^{\ell_1}w_{\ell_1}, \dots, \varepsilon\sum_{\ell_j=0}^{n}\varepsilon^{\ell_j}w_{\ell_j}\right)&,\end{aligned}$$

where G_n is such that we can decompose $\mathbb{R}$ as a finite, disjoint union of intervals J_q, $1 \leqslant q \leqslant Q$, independent of n, such that

$$\|\mathbf{1}_{t\in J_q} G_n\left(\widetilde{w}_n^\varepsilon\right)\|_F \leqslant \frac{1}{2}\|\mathbf{1}_{t\in J_q}\widetilde{w}_n^\varepsilon\|_F.$$

We infer

$$\widetilde{w}_n^\varepsilon = G_n\left(\widetilde{w}_n^\varepsilon\right) + \sum_{k=n+1}^{p-1+pn}\varepsilon^{1+k}\sum_{j=1}^{p}\sum_{\substack{j+\ell_1+\dots+\ell_j=k+1\\ 0\leqslant\ell_i\leqslant n}} N_j(\underline{u}, w_{\ell_1},\dots,w_{\ell_j}).$$

Using (2.8) and summing over the intervals J_q, we conclude

$$\|\widetilde{w}_n^\varepsilon\|_F = \mathcal{O}\left((\varepsilon\Lambda)^{n+2}\right).$$

Since $\varepsilon\Lambda < 1$ in order for all the above estimates to hold, Lemma 2.7 follows from uniqueness for (2.12) in F, which in turn is a consequence of $(H2)$. $\square$

This result allows us to infer the analyticity of the scattering operator, as shown in the following lemma.

Lemma 2.9. *Let the assumptions $(H1)$ and $(H2)$ be satisfied. Assume furthermore that $U(\cdot)$ is uniformly continuous in D. Then $U(-t)\underline{u}(t)$ converges to a limit $\underline{u}_+$ in D as $t \to +\infty$, and for all $k \geqslant 0$, $U(-t)w_k(t)$ has a limit in D, denoted by w_k^+. Moreover, for ε sufficiently small, the series $\sum_{k\in\mathbb{N}}\varepsilon^k w_k^+$ converges normally in D and the function*

$$u_+^\varepsilon := \underline{u}_+ + \varepsilon\sum_{k\in\mathbb{N}}\varepsilon^k w_k^+$$

is the limit of $U(-t)u^\varepsilon(t)$ *in* D *as* $t \to +\infty$.

In particular, the scattering operator is analytic from D *to* D.

Proof. Let us start by proving the existence of $\underline{u}_+$. We have

$$\begin{aligned}
\|U(-t_2)\underline{u}(t_2) - U(-t_1)\underline{u}(t_1)\|_D &= \left\| \int_{t_1}^{t_2} U(-s)\Phi(\underline{u})\, ds \right\|_D \\
&= \left\| \mathbf{1}_{[t_1,t_2]} U(-t) N(\underline{u}) \right\|_D \\
&\lesssim \left\| \mathbf{1}_{[t_1,t_2]} N(\underline{u}) \right\|_{F_1} \leqslant \left\| \mathbf{1}_{[t_1,t_2]} N(\underline{u}) \right\|_F \\
&\lesssim \left\| \mathbf{1}_{[t_1,t_2]} \underline{u} \right\|_{F_2}^{\delta} \|\underline{u}\|_F^{p-\delta},
\end{aligned}$$

by assumption $(H2)$. We conclude by the fact that the right-hand side goes to zero as t_1, t_2 go to infinity.

Now we prove the result on $U(-t)w_k(t)$ by induction on k. For $k=0$ we have, in the same fashion as above,

$$\begin{aligned}
\|U(-t_2)w_0(t_2) - U(-t_1)w_0(t_1)\|_D &= \left\| \mathbf{1}_{[t_1,t_2]} U(-t) N_1(\underline{u}, w_0) \right\|_D \\
&\lesssim \left\| \mathbf{1}_{[t_1,t_2]} N_1(\underline{u}, w_0) \right\|_F \\
&\lesssim \left\| \mathbf{1}_{[t_1,t_2]} \underline{u} \right\|_{F_2}^{\delta} \|\underline{u}\|_F^{p-1-\delta} \|w_0\|_F,
\end{aligned}$$

since w_0 belongs to F due to (2.11). We conclude as above.

Now suppose that for $m \geqslant 1$ and for all $0 \leqslant \ell \leqslant m-1$, $U(-t)w_\ell(t)$ has a limit. We prove the result for $U(-t)w_m(t)$. We have as above

$$\begin{aligned}
&\|U(-t_2)w_m(t_2) - U(-t_1)w_m(t_1)\|_D \leqslant \\
&\qquad \leqslant \sum_{j=1}^{p} \sum_{\substack{j+\ell_1+\cdots+\ell_j=m+1 \\ \ell_i \geqslant 0}} \left\| \mathbf{1}_{[t_1,t_2]} U(-t) N_j(\underline{u}, w_{\ell_1}, \dots, w_{\ell_j}) \right\|_D \\
&\qquad \lesssim \sum_{j=1}^{p} \sum_{\substack{j+\ell_1+\cdots+\ell_j=m+1 \\ \ell_i \geqslant 0}} \left\| \mathbf{1}_{[t_1,t_2]} N_j(\underline{u}, w_{\ell_1}, \dots, w_{\ell_j}) \right\|_F \\
&\qquad \lesssim \sum_{j=1}^{p-1} \sum_{\substack{j+\ell_1+\cdots+\ell_j=m+1 \\ \ell_i \geqslant 0}} \left\| \mathbf{1}_{[t_1,t_2]} \underline{u} \right\|_{F_2}^{\delta} \|\underline{u}\|_F^{p-\delta-j} \prod_{k=1}^{j} \|w_{\ell_k}\|_F \\
&\qquad + \sum_{\substack{p+\ell_1+\cdots+\ell_p=m+1 \\ \ell_i \geqslant 0}} \sum_{k=1}^{p} \left\| \mathbf{1}_{[t_1,t_2]} u_{\ell_k} \right\|_{F_2}^{\delta} \|u_{\ell_k}\|_F^{1-\delta} \prod_{k' \neq k} \|w_{\ell_{k'}}\|_F.
\end{aligned}$$

The result follows as previously.

The convergence of the series defining u_+^ε is due to Lemma 2.7, and Lemma 2.9 follows directly. □

2.3. **An easy and useful adaptation.** For nonlinear Schrödinger and wave equations, Lemmas 2.7 and 2.9 are well adapted to study the wave and scattering operators in energy spaces. On the other hand, as recalled in the introduction, weighted Sobolev spaces are very useful in scattering theory for these equations. Typically, for the nonlinear Schrödinger equation, the natural energy space is $H^1(\mathbb{R}^n)$, but more results concerning scattering are available in Σ, defined in the introduction. In the case of the energy space H^1, we will see that the natural choice for the space F is

$$F = C \cap L^\infty\left(\mathbb{R}; H^1(\mathbb{R}^n)\right) \cap L^{\frac{4p+4}{n(p-1)}}\left(\mathbb{R}; W^{1,p+1}(\mathbb{R}^n)\right),$$

which is of the form considered in §2.2, with $X = W^{1,p+1}(\mathbb{R}^n)$. When working on Σ, the natural choice for F is

$$\widetilde{F} = F \cap \left\{ f \in C(\mathbb{R}; \Sigma),\ J(t)f \in L^{\frac{4p+4}{n(p-1)}}\left(\mathbb{R}; L^{p+1}(\mathbb{R}^n)\right)\right\},$$

where $J(t) = x + it\nabla$ is the Galilean operator. It satisfies the important property $J(t) = U(t)xU(-t)$. The situation is fairly similar in the case of the nonlinear wave equation.

It is therefore natural to adapt the framework of §2.2. For the same spaces D and F, introduce

$$\widetilde{F} = F \cap F_3, \quad \text{where } \|f\|_{F_3} = \|Jf\|_{L^\infty(\mathbb{R};E)} + \|Jf\|_{L^q(\mathbb{R};Y)},$$

for some Banach spaces E and Y, and some operator J depending on time. Define the space $\widetilde{D}$ and $\widetilde{F}_2$ by their norms

$$\|g\|_{\widetilde{D}} = \|g\|_D + \|J(0)g\|_E \quad ; \quad \|f\|_{\widetilde{F}_2} = \|f\|_{F_2} + \|Jf\|_{L^q(\mathbb{R};Y)}.$$

It is easy to check that Lemmas 2.7 and 2.9 remain valid if F is replaced by $\widetilde{F}$, provided that (H1) and (H2) are replaced by:

$$(\widetilde{H}1) \qquad \exists C_0, \quad \forall g \in \widetilde{D}, \quad \|U(\cdot)g\|_{\widetilde{F}} \leqslant C_0\|g\|_{\widetilde{D}}.$$

and

Assumption $(\widetilde{H}2)$. There exists $\delta, C > 0$ such that for all $\underline{u}, u_1, \dots, u_j \in \widetilde{F}$ and for all I interval in $\mathbb{R}$, we have:

$$\|\mathbf{1}_{t\in I}N_j(\underline{u}, u_1, \dots, u_j)\|_{\widetilde{F}} \leqslant C\|\mathbf{1}_{t\in I}\underline{u}\|_{\widetilde{F}_2}^{\delta}\|\underline{u}\|_{\widetilde{F}}^{p-\delta-j}\prod_{\ell=1}^{j}\|u_\ell\|_{\widetilde{F}} \text{ if } j \leqslant p-1,$$

$$\|\mathbf{1}_{t\in I}N_p(u_1, \dots, u_p)\|_{\widetilde{F}} \leqslant C\sum_{\ell=1}^{p}\|\mathbf{1}_{t\in I}u_\ell\|_{\widetilde{F}_2}^{\delta}\|u_\ell\|_{\widetilde{F}}^{1-\delta}\prod_{\ell'\neq\ell}^{p}\|u_{\ell'}\|_{\widetilde{F}}.$$

In the applications, we shall also use the following lemma, whose proof follows the same lines as the proofs of Lemmas 2.7 and 2.9, and is left out.

Lemma 2.10. *Let $\underline{u} \in \widetilde{F}$ solve* (2.1) *with initial data $\underline{u}_0 \in \widetilde{D}$, and let u_0 be a given function in $\widetilde{D}$, with $\|u_0\|_{\widetilde{D}} \leqslant M$. Assume $(\widetilde{H}1)$ and $(\widetilde{H}2)$ hold. Then there exists $\varepsilon_0 = \varepsilon_0(\|\underline{u}\|_F, M) > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, the series $\sum_{k\in\mathbb{N}} \varepsilon^k w_k$ converges normally in $\widetilde{F}$, and*

$$u^\varepsilon := \underline{u} + \varepsilon\sum_{k\in\mathbb{N}}\varepsilon^k w_k \quad \textit{solves:} \quad u^\varepsilon(t) = U(t)(\underline{u}_0 + \varepsilon u_0) + N(u^\varepsilon)(t).$$

Assume furthermore that $U(\cdot)$ is uniformly continuous in $\widetilde{D}$. Then $U(-t)\underline{u}(t)$ converges to a limit $\underline{u}_+$ in $\widetilde{D}$ as $t \to +\infty$, and for all $k \geqslant 0$, $U(-t)w_k(t)$ converges to w_k^+ in $\widetilde{D}$. Moreover, for ε sufficiently small, the series $\sum_{k\in\mathbb{N}}\varepsilon^k w_k^+$

converges in $\widetilde{D}$ and the function

$$u_+^\varepsilon := \underline{u}_+ + \varepsilon \sum_{k\in\mathbb{N}} \varepsilon^k w_k^+$$

is the limit of $U(-t)u^\varepsilon(t)$ in $\widetilde{D}$ as $t \to +\infty$.

In particular, the scattering operator is analytic from $\widetilde{D}$ to $\widetilde{D}$.

3. Application to semilinear dispersive equations

3.1. **The Schrödinger equation.**

3.1.1. *General presentation.* We consider the nonlinear Schrödinger equation with gauge invariant nonlinearity presented in the introduction:

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n. \tag{3.1}$$

In order for the nonlinearity to be analytic, we assume that p is an odd integer, with $p \geqslant 3$. Note that compared to Eq. (1.1), we have imposed the value $\lambda = +1$ for the coupling constant. We consider defocusing nonlinearities, for which the scattering theory is much richer than in the focusing case, where the existence of solitons and finite time blow-up phenomenon may prevent the solution u from scattering at infinity.

Two different frameworks seem particularly well suited to study scattering for (3.1): $H^1(\mathbb{R}^n)$, and

$$\Sigma = \{f \in H^1(\mathbb{R}^n), \quad x \mapsto |x|f(x) \in L^2(\mathbb{R}^n)\}.$$

We apply Lemmas 2.7 and 2.9 in the first case, and Lemma 2.10 in the second case. Note that another framework should be well suited as well, which is the L^2 case. If $p > 1+4/n$, then the nonlinearity in (3.1) is L^2-supercritical: the results of [23] show that a scattering theory in L^2 with continuous dependence on the data is hopeless. If $p < 1+4/n$, then scattering is not known at the L^2 level, and does fail if $p \leqslant 1+2/n$ ([13, 26, 61, 62]). In the L^2-critical case $p = 1+4/n$, scattering is known for small data [20]. Note that $p = 1+4/n$ is an odd integer only when $n = 1$ or 2. For $n = 1$, scattering for large L^2 data is not known so far. For $n = 2$, scattering for large L^2 *radial* data was proved in [43]. To avoid an endless numerology, we leave out the discussion on the L^2 case at this stage.

Note also that the case of non-Euclidean geometries could be considered. In [12], the existence of scattering operators was established in H^1 for solutions to the nonlinear Schrödinger equation on hyperbolic space, in space dimension three, for energy-subcritical nonlinearities: the nonlinearity is analytic if it is cubic (and only in that case, since the energy-critical case has not been treated so far). Also, from the results in [40], scattering in H^1 is available on the two-dimensional hyperbolic space. The analyticity of wave and scattering operators in these cases can then be established by the same argument as in §3.1.2 below.

3.1.2. *The case of* H^1. For $p \geqslant 1+4/n$, with $p < 1+4/(n-2)$ when $n \geqslant 3$, the existence and continuity of wave operators was established in [29]. If we assume moreover that $p > 1+4/n$, then asymptotic completeness holds: this was proved initially in [29] for $n \geqslant 3$ (see also [63] for a simplified proof), and in [51, 53] for $n = 1, 2$ (see also [19]). We assume $1+4/n < p < 1+4/(n-2)$. In order to prove the second part of Theorem 1.2 in the energy-subcritical case, it suffices to exhibit spaces D and F_2 such that $(H1)$ and $(H2)$ are satisfied. We consider the energy-critical case $p = 1+4/(n-2)$ in a different paragraph, since the proof is slightly different.

We set naturally $D = H^1(\mathbb{R}^n)$, hence $F_1 = (C \cap L^\infty)(\mathbb{R}; H^1(\mathbb{R}^n))$. The space F_2 is motivated by Strichartz estimates:

$$F_2 = L^{\frac{4p+4}{n(p-1)}}\left(\mathbb{R}; W^{1,p+1}(\mathbb{R}^n)\right).$$

Note that the pair $(q,r) = (\frac{4p+4}{n(p-1)}, p+1)$ is L^2-admissible:

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right) =: \delta(r), \quad 2 \leqslant r \leqslant \frac{2n}{n-2}, \ (n,q,r) \neq (2,2,\infty).$$

The fact that $(H1)$ is satisfied is a consequence of homogeneous Strichartz inequalities ([30, 42]). To check $(H2)$, we use inhomogeneous Strichartz inequalities, and the following algebraic lemma:

Lemma 3.1. *Let $p \geqslant 1+4/n$, with $p < 1+4/(n-2)$ if $n \geqslant 3$. Set*

$$(q,r) = \left(\frac{4p+4}{n(p-1)}, p+1\right).$$

Then (q,r) is admissible. Set

$$\theta = \frac{p+1}{p-1} \times \frac{n(p-1)-4}{n(p-1)}.$$

Then $\theta \in [0,1[$. Define $s = r = p+1$ and $k = q/(1-\theta)$. Obviously,

$$\frac{1}{s} = \frac{1-\theta}{r} + \frac{\theta}{p+1} \quad ; \quad \frac{1}{k} = \frac{1-\theta}{q} + \frac{\theta}{\infty},$$

and we have: $\dfrac{1}{r'} = \dfrac{1}{r} + \dfrac{p-1}{s}$, *and* $\dfrac{1}{q'} = \dfrac{1}{q} + \dfrac{p-1}{k}$.

Recall that the nonlinear terms N_j stem from an inhomogeneous term in integral form, (2.6). For a time interval $I \subset \mathbb{R}$, inhomogeneous Strichartz estimates yield, for $1 \leqslant j \leqslant p$,

$$\left\| \mathbf{1}_{t\in I} N_j\left(\underline{u}, u_1, \ldots, u_j\right)\right\|_{L^\infty(\mathbb{R};L^2)\cap L^q(\mathbb{R};L^r)} \leqslant C \left\| \mathbf{1}_{t\in I} \left|\underline{u}\right|^{p-j} \prod_{\ell=1}^{j} |u_\ell| \right\|_{L^{q'}(\mathbb{R};L^{r'})},$$

for some constant C independent of I, and $\underline{u}, u_1, \dots, u_j \in F$. Using Lemma 3.1, we infer, if $j \leqslant p-1$:

$$\begin{aligned}\|\mathbf{1}_{t\in I} N_j(\dots)\|_{L^\infty L^2 \cap L^q L^r} &\lesssim \|\mathbf{1}_{t\in I}\underline{u}\|_{L^q L^r} \|\mathbf{1}_{t\in I}\underline{u}\|^{p-1-j}_{L^k L^s} \prod_{\ell=1}^{j} \|\mathbf{1}_{t\in I} u_\ell\|_{L^k L^s} \\ &\lesssim \|\mathbf{1}_{t\in I}\underline{u}\|_{L^q L^r} \|\underline{u}\|^{(1-\theta)(p-1-j)}_{L^q L^r} \|\underline{u}\|^{\theta(p-1-j)}_{L^\infty L^{p+1}} \times \\ &\quad \times \prod_{\ell=1}^{j} \|u_\ell\|^{1-\theta}_{L^q L^r} \|u_\ell\|^{\theta}_{L^\infty L^{p+1}}\end{aligned}$$

Using the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$, we deduce:

$$\|\mathbf{1}_{t\in I} N_j(\dots)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \|\mathbf{1}_{t\in I}\underline{u}\|_{L^q L^r} \|\underline{u}\|^{p-1-j}_{F} \prod_{\ell=1}^{j} \|u_\ell\|_F .$$

The estimate for N_p in $L^\infty L^2 \cap L^q L^r$ follows from the same computation. To estimate N_j in $L^\infty H^1 \cap L^q W^{1,r}$, we mimic the above computation. To simplify the presentation, and to explain why Assumption $(H2)$ is stated in such an apparently intricate way, we consider only the case $j=1$. All the other cases can be deduced in the same fashion. We have obviously

$$|\mathbf{1}_{t\in I} \nabla N_1(\underline{u}, u_1)| \lesssim \left|\mathbf{1}_{t\in I}\underline{u}^{p-2} u_1 \nabla \underline{u}\right| + \left|\mathbf{1}_{t\in I}\underline{u}^{p-1}\nabla u_1\right| .$$

Proceeding as above, we consider the $L^\infty L^2 \cap L^q L^r$ norm, and use Hölder's inequality, as suggested by Lemma 3.1. However, we do not have the same room to balance the different Lebesgue's norms: we do not want to use Sobolev embedding to control the derivatives. We find

$$\begin{aligned}\|\mathbf{1}_{t\in I} \nabla N_1(\underline{u}, u_1)\|_{L^\infty L^2 \cap L^q L^r} &\lesssim \|\nabla \underline{u}\|_{L^q L^r} \|\mathbf{1}_{t\in I}\underline{u}\|^{p-2}_{L^k L^s} \|\mathbf{1}_{t\in I} u_1\|_{L^k L^s} \\ &\quad + \|\nabla u_1\|_{L^q L^r} \|\mathbf{1}_{t\in I}\underline{u}\|^{p-1}_{L^k L^s} \\ &\lesssim \|\underline{u}\|_F \|\mathbf{1}_{t\in I}\underline{u}\|^{(1-\theta)(p-2)}_{L^q L^r} \|\underline{u}\|^{\theta(p-2)}_{L^\infty L^{p+1}} \|u_1\|_F \\ &\quad + \|u_1\|_F \|\mathbf{1}_{t\in I}\underline{u}\|^{(1-\theta)(p-1)}_{L^q L^r} \|\underline{u}\|^{\theta(p-1)}_{L^\infty L^{p+1}} \\ &\lesssim \|\mathbf{1}_{t\in I}\underline{u}\|^{1-\theta}_{L^q L^r} \|\underline{u}\|^{p+\theta-2}_{F} \|u_1\|_F ,\end{aligned}$$

where we have used the same estimates as above (recall that $p \geqslant 3$). Therefore, Assumption $(H2)$ is satisfied, with $\delta = 1-\theta$. Note that $\delta > 0$ because we consider the energy-subcritical case, $p < 1 + 4/(n-2)$.

Therefore, we can apply Lemmas 2.7 and 2.9 with F as above. This yields the second part of Theorem 1.2, except for the energy-critical case. Note that in the following two cases:

- $n = 1$ and $p = 5$ (quintic nonlinearity),
- $n = 2$ and $p = 3$ (cubic nonlinearity),

which are L^2 critical $p = 1 + 4/n$, Lemma 2.7 shows that the wave operators are analytic on $H^1(\mathbb{R}^n)$. However, scattering in the energy space for arbitrary data is not known in these cases.

3.1.3. *The case of* Σ. To overcome the drawback mentioned at the end of the previous paragraph, we shall consider the weighted Sobolev space Σ. Generally speaking, working in Σ makes it possible to decrease the admissible values for p in order to have scattering, from $p > 1 + 4/n$, to $p \geqslant p_0(n)$, for some $1 + 2/n < p_0(n) < 1 + 4/n$; see [22, 27, 36, 54]. However, the gain in the present context is rather weak, since we consider only integer values for p: the gain corresponds exactly to the two cases pointed out above.

As suggested in §2.3, we consider the space

$$\widetilde{F} = F \cap \left\{ f \in C(\mathbb{R};\Sigma),\ J(t)f \in L^{\frac{4p+4}{n(p-1)}}\left(\mathbb{R}; L^{p+1}(\mathbb{R}^n)\right) \right\},$$

where $J(t) = x + it\nabla$, and F was defined in the previous paragraph. We can then mimic the above computation, in order to apply Lemma 2.10. We recall two important properties of the operator J which make it possible to check Assumptions $(\widetilde{H}1)$ and $(\widetilde{H}2)$:

- It commutes with the linear Schrödinger group: $J(t) = U(t)xU(-t)$.
- It acts on gauge invariant nonlinearities like a derivative, since

$$J(t) = ite^{i|x|^2/(2t)}\nabla\left(e^{-i|x|^2/(2t)}\cdot\right), \quad \forall t \neq 0.$$

Lemma 2.10 and the results of [27] yield Theorem 1.2 in all the cases, but the energy critical one, which is considered in the next paragraph.

3.1.4. *The energy-critical case.* To complete the proof of Theorem 1.2, two cases remain, which correspond to the case $p = 1 + 4/(n-2)$:

- $n = 3$ and $p = 5$.
- $n = 4$ and $p = 3$.

Global existence and scattering for arbitrary data in $H^1(\mathbb{R}^n)$ were established in [24] and [57], respectively. A crucial tool in the energy critical case is the existence of Strichartz estimates for $\dot{H}^1$-admissible pairs, as opposed to the notion of L^2-admissible pairs used above. It is fairly natural that our definition for F is adapted in view of this notion. Recall that for $n \geqslant 3$, a pair (q, r) is $\dot{H}^1$-admissible if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - 1.$$

Denote

$$\gamma_0 = 2 + \frac{4}{n} \quad \text{and} \quad \gamma_1 = 2 + \frac{8}{n-2}.$$

The pair (γ_0, γ_0) is L^2-admissible, and (γ_1, γ_1) is $\dot{H}^1$-admissible. We set

$$F = F_1 \cap F_2, \text{ with } F_1 = (C \cap L^\infty)\left(\mathbb{R}; H^1(\mathbb{R}^n)\right), \text{ and}$$
$$F_2 = L^{\gamma_0}\left(\mathbb{R}; W^{1,\gamma_0}(\mathbb{R}^n)\right) \cap L^{\gamma_1}(\mathbb{R} \times \mathbb{R}^n).$$

With such a space F, Assumption $(H1)$ is satisfied, thanks to Strichartz estimates, along with the Sobolev embedding $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{R}^n)$. To check that Assumption $(H2)$ is satisfied as well, we distinguish the two cases we consider, for a more convenient numerology.

The quintic case, with $n = 3$. In this case, we have $\gamma_0 = 10/3$ and $\gamma_1 = 10$. For $u_1, \dots, u_5 \in F$, we have, for $k = 0$ or 1, thanks to Strichartz estimates and Hölder's inequality:

$$\begin{aligned}
\Big\|\nabla^k \int_{t_0}^t U(t-s)\left(u_1 \times \dots \times u_5\right)(s)ds\Big\|_{L^\infty(I;L^2)\cap L^{10/3}(I\times\mathbb{R}^n)} \\
\lesssim \Big\|\nabla^k\left(u_1 \times \dots \times u_5\right)\Big\|_{L^{10/7}(I\times\mathbb{R}^n)} \\
\lesssim \sum_{j=1}^5 \Big\|\nabla^k u_j\Big\|_{L^{10/3}(I\times\mathbb{R}^n)} \prod_{\ell\neq j}\|u_\ell\|_{L^{10}(I\times\mathbb{R}^n)} \lesssim \prod_{j=1}^5 \|\mathbf{1}_{t\in I}u_j\|_{F_2}.
\end{aligned}$$

We also have, in view of Sobolev embedding,

$$\begin{aligned}
\Big\|\int_{t_0}^t U(t-s)\left(u_1 \times \dots \times u_5\right)(s)ds\Big\|_{L^{10}(I\times\mathbb{R}^n)} \\
\lesssim \Big\|\int_{t_0}^t U(t-s)\left(u_1 \times \dots \times u_5\right)(s)ds\Big\|_{L^{10}(I;W^{1,30/13})} \\
\lesssim \sum_{k=0}^1 \Big\|\nabla^k\left(u_1 \times \dots \times u_5\right)\Big\|_{L^{10/7}(I\times\mathbb{R}^n)},
\end{aligned}$$

thanks to Strichartz estimates. Using the above computation, we infer that Assumption $(H2)$ is satisfied.

The cubic case, with $n = 4$. In this case we have $\gamma_0 = 3$ and $\gamma_1 = 6$. For $u_1, u_2, u_3 \in F$, we have, for $k = 0$ or 1, thanks to Strichartz estimates and Hölder's inequality:

$$\begin{aligned}
\Big\|\mathbf{1}_{t\in I}\nabla^k \int_{t_0}^t U(t-s)\left(u_1u_2u_3\right)(s)ds\Big\|_{L^\infty_t L^2_x\cap L^3_{t,x}} \lesssim \Big\|\nabla^k\left(u_1u_2u_3\right)\Big\|_{L^{3/2}(I\times\mathbb{R}^n)} \\
\lesssim \sum_{j=1}^3 \Big\|\nabla^k u_j\Big\|_{L^3(I\times\mathbb{R}^n)} \prod_{\ell\neq j}\|u_\ell\|_{L^6(I\times\mathbb{R}^n)} \lesssim \prod_{j=1}^3 \|\mathbf{1}_{t\in I}u_j\|_{F_2}.
\end{aligned}$$

We also have, in view of Sobolev embedding,

$$\begin{aligned}
\Big\|\int_{t_0}^t U(t-s)\left(u_1u_2u_3\right)(s)ds\Big\|_{L^6(I\times\mathbb{R}^n)} \\
\lesssim \Big\|\int_{t_0}^t U(t-s)\left(u_1u_2u_3\right)(s)ds\Big\|_{L^6(I;W^{1,12/5})} \lesssim \sum_{k=0}^1 \Big\|\nabla^k\left(u_1u_2u_3\right)\Big\|_{L^3(I\times\mathbb{R}^n)},
\end{aligned}$$

thanks to Strichartz estimates. Using the above computation, we infer that Assumption $(H2)$ is satisfied.

Finally, it is easily checked that we can replace H^1 with Σ, as in the previous paragraph. This completes the proof of Theorem 1.2.

Remark 3.2. At the level of H^1, it is possible to have a unified presentation, that is, without distinguishing the H^1-subcritical and H^1-critical cases. The price to pay consists in considering Besov spaces for the definition of F_2, instead of Sobolev spaces. We have chosen to work in Sobolev for the simplicity and the explicit form of the computations. A more synthetic approach would consist in setting

$$F_2 = L^{\gamma_0}\left(\mathbb{R}; B^1_{\gamma_0,2}(\mathbb{R}^n)\right) \cap L^{\gamma_1}\left(\mathbb{R}\times\mathbb{R}^n\right),$$
$$\text{with } \gamma_0 = 2+\frac{4}{n} \text{ and } \frac{p-1}{\gamma_1}+\frac{1}{\gamma_0}=\frac{1}{\gamma_0'}.$$

Sobolev and Strichartz inequalities are replaced by

$$\|u\|_{L^\infty(\mathbb{R};H^1)} + \|u\|_{F_2} \leqslant C\left(\|u_0\|_{H^1} + \left\|i\partial_t u + \frac{1}{2}\Delta u\right\|_{L^{\gamma_0'}\left(\mathbb{R};B^1_{\gamma_0',2}(\mathbb{R}^n)\right)}\right),$$

an estimate established in [51, §3]. Note that in the energy-critical case $p = 1+\frac{4}{n-2}$, this is the estimate which we have used, up the replacing Besov spaces $B^1_{p,2}$ with $W^{1,p}$ (a modification which is non-trivial since $p\neq 2$).

3.2. **The Hartree equation.** We now consider the Hartree equation (1.2) with a defocusing nonlinearity, $\lambda = +1$, in space dimension $n \geqslant 3$:

$$i\partial_t u + \frac{1}{2}\Delta u = \left(|x|^{-\gamma} * |u|^2\right)u. \tag{3.2}$$

Note that the nonlinearity $u \mapsto \left(|x|^{-\gamma} * |u|^2\right)u$ is always a smooth homogeneous (cubic) function of u. We assume $2 \leqslant \gamma < \min(4,n)$. A complete scattering theory is available in the space Σ; see [28, 37]. If we assume moreover $\gamma > 2$, then Σ can be replaced by $H^1(\mathbb{R}^n)$; see [34, 50]. The counterpart of Lemma 3.1 is:

Lemma 3.3. *Let $n \geqslant 3$ and $2 \leqslant \gamma < \min(4,n)$. Set*

$$(q,r) = \left(\frac{8}{\gamma}, \frac{4n}{2n-\gamma}\right).$$

Then (q,r) is L^2-admissible. Set $\theta = 2 - 4/\gamma$. Then $\theta \in [0,1[$. Define $s = r$ and $k = q/(1-\theta)$. Obviously,

$$\frac{1}{s} = \frac{1-\theta}{r} + \frac{\theta}{r} \quad ; \quad \frac{1}{k} = \frac{1-\theta}{q} + \frac{\theta}{\infty},$$

and we have $s < \dfrac{2n}{n-\gamma}$, with $\dfrac{1}{r'} = \dfrac{1}{r} + \dfrac{2}{s} + \dfrac{\gamma}{n} - 1$ and $\dfrac{1}{q'} = \dfrac{1}{q} + \dfrac{2}{k}$.

We can then proceed as in the energy-subcritical case for the nonlinear Schrödinger equation (3.1), in order to prove Theorem 1.3. The only difference is the use of the Hardy–Littlewood–Sobolev inequality. Since the computations are very similar to those presented in §3.1, we shall be rather sketchy, and detail only the most important computation. We set

$$F_1 = (C\cap L^\infty)(\mathbb{R}; H^1(\mathbb{R}^n)) \quad ; \quad F_2 = L^q\left(\mathbb{R}; W^{1,r}(\mathbb{R}^n)\right),$$

where (q,r) is now given by Lemma 3.3. It follows from Strichartz estimates that $(H1)$ is satisfied. For $t \in \overline{\mathbb{R}}$ and I an interval in $\mathbb{R}$, we have, for $\ell = 0$

or 1:

$$\begin{aligned}
&\left\|\mathbf{1}_{t\in I}\nabla^\ell \int_{t_0}^t U(t-\tau)\left(\left(|x|^{-\gamma}*(u_1u_2)\right)u_3\right)(\tau)d\tau\right\|_{L^\infty_tL^2_x\cap L^q_tL^r_x}\\
&\quad\lesssim \left\|\mathbf{1}_{t\in I}\nabla^\ell\left(|x|^{-\gamma}*(u_1u_2)\right)u_3\right\|_{L^{q'}_tL^{r'}_x}\\
&\quad\lesssim \left\|\|u_1\nabla^\ell u_2\|_{L^{s/2}_x}\|u_3\|_{L^r_x}\right\|_{L^{q'}_t(I)}+\left\|\|u_2\nabla^\ell u_1\|_{L^{s/2}_x}\|u_3\|_{L^r_x}\right\|_{L^{q'}_t(I)}\\
&\quad+\left\|\|u_1u_2\|_{L^{s/2}_x}\|\nabla^\ell u_3\|_{L^r_x}\right\|_{L^{q'}_t(I)}\\
&\quad\lesssim \sum_{j=1}^3\left\|\|\nabla^\ell u_j\|_{L^r_x}\prod_{j'\neq j}\|u_{j'}\|_{L^r_x}\right\|_{L^{q'}_t(I)}
\end{aligned}$$

where we have used Hölder and Hardy–Littlewood–Sobolev inequalities in the space variable. Using Hölder's inequality in time, we can estimate each term of the above sum by:

$$\begin{aligned}
&\left\|\nabla^\ell u_j\right\|_{L^q(I;L^r)}\prod_{j'\neq j}\|u_{j'}\|_{L^k(I;L^r)}\\
&\qquad\qquad\lesssim \left\|\nabla^\ell u_j\right\|_{L^q(I;L^r)}\prod_{j'\neq j}\left(\|u_{j'}\|^{1-\theta}_{L^q(I;L^r)}\|u_{j'}\|_{L^\infty(I;L^r)}\right)\\
&\qquad\qquad\lesssim \prod_{j=1}^3\|\mathbf{1}_{t\in I}u_j\|^{1-\theta}_{F_2}\|u_j\|^\theta_F,
\end{aligned}$$

where we have used the embedding $H^1\hookrightarrow L^r$. This estimate suffices to check that Assumption $(H2)$ is satisfied (with $\delta=1-\theta>0$), hence Theorem 1.3 in the case of $H^1(\mathbb{R}^n)$. In the case of Σ (which allows to consider the value $\gamma=2$), one uses the operator $J(t)=x+it\nabla$ like in §3.1.3, to complete the proof of Theorem 1.3.

3.3. **The wave equation.** We now turn to the case of the nonlinear wave equation

$$\partial_t^2u-\Delta u+u^p=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}^n. \tag{3.3}$$

In order for the nonlinearity to be analytic, we assume that p is an integer. Moreover, for the anti-derivative of the nonlinearity to have a constant sign, we need to assume that p is odd; without this assumption, scattering for arbitrary large data does not hold.

The existence of wave and scattering operators in

$$\Sigma_2=\{(f,g)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n),\quad x\mapsto|x|\nabla f(x),x\mapsto|x|g(x)\in L^2(\mathbb{R}^n)\}$$

was established in [31], under the assumption

$$1+\frac{4}{n-1}\leqslant p<1+\frac{4}{n-2}.$$

As a matter of fact, some values for $p<1+4/(n-1)$ are also allowed there. See also [6] and [39] for $n=p=3$. With these results, we could certainly prove that the wave and scattering operators are analytic from Σ_2 to Σ_2, for $2\leqslant n\leqslant 4$ and

- $p \geqslant 5$ if $n = 2$.
- $p = 3$ or 5 if $n = 3$.
- $p = 3$ if $n = 4$.

We leave out the discussion at this stage, since the estimates based on the conformal decay are fairly long to write.

The existence of wave and scattering operators in $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ was established in [9, 41, 59, 60] for the energy-critical case

$$p = 1 + \frac{4}{n-2}, \quad n = 3, 4.$$

(The space dimensions 3 and 4 are the only ones for which the energy-critical nonlinearity corresponds to an odd integer p.) As stated in Theorem 1.4, we shall content ourselves with these two cases. Note also that from [32], the existence of scattering operators in the energy space is known for energy-subcritical nonlinearities. However, this range for p does not include odd integers, and we are left with the above two cases. Also, if we considered only small data scattering, then more results would be available. We choose not to distinguish too many cases, and restrict our attention to the framework of Theorem 1.4.

Naturally, we have $D = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, and

$$F_1 = (C \cap L^\infty)(\mathbb{R}; \dot{H}^1(\mathbb{R}^n)) \times (C \cap L^\infty)(\mathbb{R}; L^2(\mathbb{R}^n)).$$

As in the case of the Schrödinger equations studied above, the space F_2 is defined using Strichartz estimates: we set

$$F_2 = \begin{cases} L^5\left(\mathbb{R}; L^{10}(\mathbb{R}^3)\right) \times L^\infty\left(\mathbb{R}; L^2(\mathbb{R}^3)\right) & \text{if } n = 3, \\ L^3\left(\mathbb{R}; L^6(\mathbb{R}^4)\right) \times L^\infty\left(\mathbb{R}; L^2(\mathbb{R}^4)\right) & \text{if } n = 4. \end{cases}$$

Recall that for $n \geqslant 3$, and (q, r) satisfying

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - 1, \quad 6 \leqslant r < \infty \text{ if } n = 3, \ \frac{2n}{n-2} \leqslant r \leqslant \frac{2n+2}{n-3} \text{ if } n \geqslant 4,$$

Strichartz estimates yield (see e.g. [33, 42])

$$\begin{aligned} &\|u\|_{L^q(I;L^r)} + \|u\|_{L^\infty(I;\dot{H}^1)} + \|\partial_t u\|_{L^\infty(I;L^2)} \\ &\qquad \leqslant C_r \left(\left\| u_{|t=0} \right\|_{\dot{H}^1} + \left\| \partial_t u_{|t=0} \right\|_{L^2} + \left\| \left(\partial_t^2 - \Delta\right) u \right\|_{L^1(I;L^2)} \right), \end{aligned}$$

for some constant C_r independent of the time interval I. Note that the pairs $(5, 10)$ and $(3, 6)$ are admissible for $n = 3$ and $n = 4$, respectively.

In the case $n = 3$, and in view of Example 2.2, it is enough to control $\|u_1 u_2 u_3 u_4 u_5\|_{L^1(I;L^2)}$ by the product of the $\|u_j\|_{L^5(I;L^{10})}$, to verify Assumption $(H2)$. Such an estimate if of course trivially satisfied. Similarly, for $n = 4$, $\|u_1 u_2 u_3\|_{L^1(I;L^2)}$ is controlled by the product of the $\|u_j\|_{L^3(I;L^6)}$. Therefore, Theorem 1.4 follows from Lemmas 2.7 and 2.9.

3.4. **The Klein–Gordon equation.** We conclude with the case of the Klein–Gordon equation

$$\partial_t^2 u - \Delta u + u + u^p = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.4}$$

As above, we assume that p is an odd integer. The natural energy space is $D = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. For $n \geqslant 3$, scattering in the energy space was established in [16] for

$$1 + \frac{4}{n} < p \leqslant 1 + \frac{4}{n-1},$$

and in [30] for

$$1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}.$$

The case of the low dimensions $n = 1$ or 2 was treated by K. Nakanishi [51] (see also [53]), for $p > 1 + 4/n$. The existence of wave and scattering operators in the energy-critical case $p = 1 + 4/(n-2)$ in space dimension $n \geqslant 3$ was established in [52]. All in all, scattering in the energy space is known for $p > 1+4/n$, and $p \leqslant 1+4/(n-2)$ when $n \geqslant 3$. Such values for p corresponding to an odd integer are exactly those considered in Theorem 1.5.

As pointed out in [31], this numerology is the same as in the case of the nonlinear Schrödinger equation (3.1). The proof of Theorem 1.5 follows essentially the same lines as the proof of Theorem 1.2, up to the following adaptation. For the space F_1, we keep

$$F_1 = \left(C \cap L^\infty\right)\left(\mathbb{R}; H^1(\mathbb{R}^n)\right).$$

For the space F_2, Sobolev spaces are replaced by Besov spaces:

$$F_2 = L^{\gamma_0}\left(\mathbb{R}; B^{1/2}_{\gamma_0,2}(\mathbb{R}^n)\right) \cap L^{\gamma_1}\left(\mathbb{R}\times\mathbb{R}^n\right),$$
$$\text{with } \gamma_0 = 2 + \frac{4}{n} \text{ and } \frac{p-1}{\gamma_1} + \frac{1}{\gamma_0} = \frac{1}{\gamma_0'}.$$

Equation (3.9) in [51] yields the analogue of the estimate recalled in Remark 3.2:

$$\|u\|_{L^\infty(\mathbb{R};H^1)} + \|\partial_t u\|_{L^\infty(\mathbb{R};L^2)} + \|u\|_{F_2}$$
$$\leqslant C\left(\|u_0\|_{H^1} + \|u_1\|_{L^2} + \left\|\partial_t^2 u - \Delta u + u\right\|_{L^{\gamma_0'}\left(\mathbb{R}; B^{1/2}_{\gamma_0',2}(\mathbb{R}^n)\right)}\right),$$

The proof of Theorem 1.5 then follows the same lines as the proof of Theorem 1.2, up to the technical modifications which can be found in [51].

4. Some consequences

4.1. **Invariant skew-symmetric forms.** Let

$$\omega_{\rm wave}\left(u_1,u_2\right)(t) := \int_{\mathbb{R}^n} \left(u_1\partial_t u_2 - u_2\partial_t u_1\right)(t,x)dx. \tag{4.1}$$

It is proved in [49] that for the cubic three-dimensional Klein–Gordon equation (Eq. (3.4) with $n = p = 3$), $\omega_{\rm wave}$ induces a skew-symmetric differential form on some space F (based on the energy space), which is invariant under S. In [8], the space F was replaced by the energy space, in the small data case. Following the proof of [49], we have the following extension:

Proposition 4.1. *For $m \geqslant 0$, consider the equation (wave or Klein–Gordon)*

$$\partial_t^2 u - \Delta u + m^2 u + u^p = 0.$$

Then under the algebraic assumptions of Theorem 1.4 (case $m = 0$) or Theorem 1.5 (case $m > 0$), ω_{wave} induces a skew-symmetric differential form on the energy space, which is invariant under S.

Sketch of the proof. Since the proof follows the same lines as in [49], we shall simply recall the main steps. At least for smooth solutions, we compute

$$\frac{d}{dt}\omega_{\text{wave}}(u_1, u_2) = \int_{\mathbb{R}^n} (u_2 u_1^p - u_1 u_2^p)\, dx.$$

If u_1, u_2 and u_3 solve the above equation, then using the above relation and expanding

$$\omega_{\text{wave}}\left(u_2 - u_1, u_3 - u_1\right) = \omega_{\text{wave}}\left(u_2, u_3\right) + \omega_{\text{wave}}\left(u_1, u_2\right) - \omega_{\text{wave}}\left(u_1, u_3\right),$$

we find

$$\begin{aligned} \frac{d}{dt}\omega_{\text{wave}}\left(u_2 - u_1, u_3 - u_1\right) = &\int \left(\left(u_2^{p-1} - u_3^{p-1}\right)(u_2 - u_1)u_3\right) dx \\ &+ \int \left(\left(u_2^{p-1} - u_1^{p-1}\right)(u_3 - u_2)u_1\right) dx. \end{aligned} \tag{4.2}$$

Elementary computations show that $(u_1 - u_2)(u_1 - u_3)(u_2 - u_3)$ can be factored out in the above expression. Now let u_-, v_- and w_- be in the energy space (whose definition varies whether $m = 0$ or $m > 0$). In (4.2), we consider u_1, u_2 and u_3 with asymptotic states as $t \to -\infty$ given by u_-, $u_- + \varepsilon v_-$ and $u_- + \varepsilon w_-$, respectively. The results of Section 2 show that the image of v_- under $dS(u_-)$ is v_+, which is the asymptotic state as $t \to +\infty$ of v, satisfying

$$\partial_t^2 v - \Delta v + m^2 v + p u^{p-1} v = 0,$$

with asymptotic state v_- as $t \to -\infty$ ($v_+ = v_-$ if $u \equiv 0$: S is almost the identity near the origin; v_+ is implicit otherwise, see §2.1). Integrating (4.2) over all t, we get:

$$\omega_{\text{wave}}\left((u_2 - u_1)_+, (u_3 - u_1)_+\right) - \omega_{\text{wave}}\left(\varepsilon v_-, \varepsilon w_-\right) = \mathcal{O}\left(\varepsilon^3\right),$$

from the factorization mentioned above. Simplifying by ε^2, the result follows by letting $\varepsilon \to 0$. □

In the case of the Schrödinger operator, introduce

$$\omega_{\text{Schröd}}\left(u_1, u_2\right)(t) = \operatorname{Im} \int_{\mathbb{R}^n} \left(\overline{u}_1 u_2\right)(t, x) dx.$$

Like above, if u_1 and u_2 solve

$$i\partial_t u_j + \frac{1}{2}\Delta u_j = F_j,$$

then we have:

$$\frac{d}{dt}\omega_{\text{Schröd}}\left(u_1, u_2\right) = \operatorname{Re} \int_{\mathbb{R}^n} \left(\overline{F}_1 u_2 - \overline{u}_1 F_2\right).$$

If u_1, u_2 and u_3 solve (3.1), we find:

$$\frac{d}{dt}\omega_{\text{Schröd}}\left(u_2-u_1,u_3-u_1\right)=\int\left(|u_2|^{p-1}-|u_3|^{p-1}\right)\operatorname{Re}(u_2-u_1)\overline{u}_3 \\ +\int\left(|u_2|^{p-1}-|u_1|^{p-1}\right)\operatorname{Re}(u_3-u_2)\overline{u}_1.$$

Viewing the right hand side as a polynomial in three unknowns u_1, u_2 and u_3, we note that it is zero for $u_1=u_2$, $u_3=u_1$ and $u_2=u_3$. We can then use the same argument as above, to claim that it yields a contribution of order $\mathcal{O}(\varepsilon^3)$. Proceeding as above, we have:

Proposition 4.2. *Consider the equation*

$$i\partial_t u+\frac{1}{2}\Delta u=|u|^{p-1}u.$$

Under the algebraic assumptions of Theorem 1.2, $\omega_{Schröd}$ induces a skew-symmetric differential form on $H^1(\mathbb{R}^n)$ (or Σ), which is invariant under S, the scattering operator associated to the above equation.

Finally, if u_1, u_2 and u_3 solve

$$i\partial_t u_j+\frac{1}{2}\Delta u_j=\left(V*|u_j|^2\right)u_j,$$

then we find

$$\frac{d}{dt}\omega_{\text{Schröd}}\left(u_2-u_1,u_3-u_1\right)=\int\left(V*\left(|u_2|^2-|u_3|^2\right)\right)\operatorname{Re}(u_2-u_1)\overline{u}_3 \\ +\int\left(V*\left(|u_2|^2-|u_1|^2\right)\right)\operatorname{Re}(u_3-u_2)\overline{u}_1.$$

Proposition 4.3. *Consider the equation*

$$i\partial_t u+\frac{1}{2}\Delta u=\left(|x|^{-\gamma}*|u|^2\right)u.$$

Under the algebraic assumptions of Theorem 1.3, $\omega_{Schröd}$ induces a skew-symmetric differential form on $H^1(\mathbb{R}^n)$ (or Σ), which is invariant under S, the scattering operator associated to the above equation.

4.2. **Infinitely many conserved quantities.** In [5, 8], the authors consider the Klein-Gordon equations (1.4) with $p=3$, and prove that the analyticity of the scattering operator (which at the time was only known for small data) implies the existence of a complete set of conserved quantities with vanishing Poisson brackets. The proof of [8] relies upon the construction of invariant skew-symmetric forms, as in the previous section. Once the form ω_{wave} is known, one can construct explicitly a complete set of integrals of motion F_j, with vanishing Poisson brackets. The statement is given below, in all the cases studied in the paper. We refer to [8] for the proof of the result, which can be directly adapted to the skew-symmetric form $\omega_{\text{Schröd}}$.

Proposition 4.4. *For each of the equations (1.1) to (1.4) considered in this paper, and under the algebraic assumptions of Theorems 1.2 to 1.5 respectively there is a family F_j of analytic functionals acting from the space* *of initial data D into $\mathbb{R}$, invariant under the nonlinear evolution, and such that there is a vector field v_j in D such that*

$$dF_j = \omega(v_j, \cdot)$$

where ω denotes respectively $\omega_{Schröd}$ and $\omega_{\rm wave}$. Moreover, generically in u, for any couple of vector fields (v,w) in $T_u D$ such that $dF_j v = dF_j w = 0$, we have $\omega(v,w) = 0$.

This result can be understood as the existence of a Birkhoff normal form (see e.g. [10, 35] for a general definition and a presentation of results). However, for nonlinear equations, Birkhoff normal forms are usually employed to establish long time existence results (see e.g. [15, 11]), whereas in our case, they come as a consequence of asymptotic properties of solutions which are already known to exist globally.

4.3. **Inverse scattering.** As noticed in [49, Theorem 2], knowing the scattering operator near the origin for a nonlinear equation with analytic nonlinearity suffices to determine the nonlinearity, since the coefficients of its Taylor series can be computed by induction.

In [58], the first term of the asymptotic expansion of the scattering operator is shown to fully determine a nonlocal nonlinearity whose form is known in advance (Hartree type nonlinearity). This approach is applied in the Schrödinger case, as well as in the Klein–Gordon case. In that case, the nonlinearity need not be analytic, and only the first nontrivial term of the asymptotic expansion of S near the origin is needed. Typically, in the same spirit, consider the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u, \tag{4.3}$$

with $\lambda \in \mathbb{R}$ (possibly negative), $p \geqslant 1 + 4/n$ and $p \leqslant 1 + 4/(n-2)$ if $n \geqslant 3$, not necessarily an integer. For small data, solutions to (4.3) are global in time, and admit scattering states. To see this, recall that the nonlinearity in (4.3) is H^s-critical, with

$$s = \frac{n}{2} - \frac{2}{p-1} \geqslant 0.$$

In the small data case, Strichartz and Sobolev inequalities show that global existence and scattering follow from a simple bootstrap argument (see e.g. [20] in the case of $s = 0$, [21] in the case $s > 0$). In addition, we have

$$W_\pm(\varepsilon\phi) = \varepsilon\phi + i\lambda\varepsilon^p \int_0^{\pm\infty} e^{-i\frac{t}{2}\Delta}\left(\left|e^{i\frac{t}{2}\Delta}\phi\right|^{p-1} e^{i\frac{t}{2}\Delta}\phi\right) dt + \mathcal{O}_{H^s}\left(\varepsilon^{2p-1}\right),$$

hence

$$S(\varepsilon\phi) = \varepsilon\phi - i\lambda\varepsilon^p \int_{-\infty}^{+\infty} e^{-i\frac{t}{2}\Delta}\left(\left|e^{i\frac{t}{2}\Delta}\phi\right|^{p-1} e^{i\frac{t}{2}\Delta}\phi\right) dt + \mathcal{O}_{H^s}\left(\varepsilon^{2p-1}\right).$$

See [18] for the proof in the case $s = 0$. The proof for $s > 0$ follow the same lines, up to the modifications which can be found in [21]. Loosely speaking, the leading order term of $S(\varepsilon\phi) - \varepsilon\phi$ suffices to determine λ and p. For instance,

$$p = \lim_{\varepsilon\to 0} \frac{\log\|S(\varepsilon\phi) - \varepsilon\phi\|_{H^s}}{\log\varepsilon},$$

for ϕ a Gaussian function, so that the term in ε^p cannot be zero.

4.4. **On the complete integrability.** When speaking of complete integrability, one has to be rather cautious: several notions are present in the literature [4, 65]. The weakest definition (which is in fact useful mainly in a finite dimensional situation) consists in saying that there exists as many conserved quantities as the number of degrees of freedom (infinitely many in infinite dimensional situations), with vanishing Poisson brackets; this corresponds to the discussion in Section 4.2 above. One can observe that those conserved quantities may not be relevant in terms of Sobolev norms (see for example [14]). In the Hamiltonian case, the quantities are the Hamiltonian and first integrals; see e.g. [1, 2, 3]. At a higher (in the infinite dimensional case) level of precision, there may exist a nonlinear change of variables which makes the original equation linear. This is typically the case of one-dimensional Schrödinger equations with cubic nonlinearity [66], and is related to the existence of Lax pairs [46]. The strongest notion of integrability consists in trivializing the equation on some Lie algebra; see e.g. [38].

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(R. Carles) CNRS & Université Montpellier 2, Mathématiques, CC 051, Place Eugène Bataillon, 34095 Montpellier cedex 5, France
E-mail address: Remi.Carles@math.cnrs.fr

(I. Gallagher) Institut de Mathématiques de Jussieu UMR 7586, Université Paris VII, 175, rue du Chevaleret, 75013 Paris, France
E-mail address: Isabelle.Gallagher@math.jussieu.fr